A counterexample to one property from generalized hukuhara differentiability defined by stefanini-bede

Investigación terminada

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ABSTRACT

In [2] was defined one notion of differentiability more general which was given in [1]. We give an counterexample to one property of Generalized Hukuhara differentiability given in [2]. This has implications on fuzzy differential equations.

Index Terms—Fuzzy Logic, Hukuhara Differentiability, **G**eneralized Differentiability

Definition 1.1. *[3] Let T = (a,b) and let f be a set-valved mapping* $f : T \to I$. *f is said H-differentiable at a point x* ∈ *T if for h > 0 enough small, the differences f (x+h)* Θ_H *f (x), f (x)* Θ_H *f (x-h) exist, and there also exists f '(x)* ∈ *(I such that*.

I. PRELIMINARY

SO we denote *I* the collection of all nonempty-closed intervals of ℝ Of course, the elements of *I* are convex on ℝ. If $A, B \in \mathbb{R}$, then the addition and the scalar multiplication in *I* are defined in usual way as

$$A + B = \{a + b | a \in A, b \in B\}, \ \leftthreetimes A = \leftthreetimes a | a \in A\}. \qquad (1)$$

The Hausdorff metric d_H on *I* can be defined as

$$d_H\,(A, B) = max\{|a^L - b^L|, |a^U - b^U|\},$$

were $A = [a^L, a^U], B = [b^L, b^U] \in I.$

It is known that the space *I* is not a linear space, since is does not contain inverse elements. In order to overcome this difficulty some alternatives have been proposed. In fact, initially the Hukuhara difference (H-difference) in *I* was introduced. The H-difference, denoted by $A \Theta_H B$, is defined as

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$$A\ \Theta_H B = C \Leftrightarrow A = B + C.(2) \qquad (2)$$

Whit this definition, $A\ \Theta_H A = \{0\}$ for all $A \in I$ and $(A + B)\ \Theta_H B = A$, for all $A, B \in I$. Moreover, if there is $C \in I$ such that $A\ \Theta_H B = C$, then *C* is unique; however, the H-difference does not always exist, for example, in *I*, if $A = [2, 3]$ and $B = [5, 10]$, the difference $A\ \Theta_H B$ does not exist. In general, $A - B \neq A\ \Theta_H B$.

Based on Definition of H-difference on *I,* in [3] was defined the notion of *H*-differentiability for set-valued mappings. Let $f : T \to I$ be a valued map, then it can be written by $f(x) = [f^L(x), f^U(x)] \in T$. This is, we can define two real-valued functions based on extremes of intervals,

$$f^L(x) = (f(x))^L \quad \text{and} \quad f^U(x) = (f(x))^U. \qquad (3)$$

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$$\lim_{h \to 0^+}\left(\frac{f(x+h)\theta_H f(x)}{h}\right) = \lim_{h \to 0^+}\left(\frac{f(x)\theta_H f(x-h)}{h}\right) = f'(x), \qquad (4)$$

where the limits are taken in the metric space (I, d_H)

As pointed out in [1], the Definition 1.1 of H-differentiability of a set-valued mapping is very restrictive. To solve that, the authors of [1] introduced the notion of generalized differentiability by taking into account the lateral types of *H*-derivatives, as follows.

Definition 1.2. *[1] Let T =* (a,b) *and let* $f : T \to I$ *be a set-valved mapping. fis said to be strongly generalized differentiable at x* ∈ *T, if for h >* 0 *enough small,*

1) *there exist the differences f (x + h)* Θ_H *f (x), f (x)* Θ_H *f (x-h) and there also exists f '(x)* ∈ *I such that*

$$\lim_{h \to 0}\left(\frac{f(x+h)\theta_H f(x)}{h}\right) = \lim_{h \to 0}\left(\frac{f(x)\theta_H f(x-h)}{h}\right) = f'(x) \qquad (5)$$

or

2) *there exist the differences f (x + h) Θ_H f (x), f (x) Θ_H f (x-h) and there also exists f '(x) ∈ I such that*

$$\lim_{h\to 0}\left(\frac{f(x-h)\theta_H f(x)}{-h}\right) = \lim_{h\to 0}\left(\frac{f(x)\theta_H f(x+h)}{-h}\right) = f'(x) \quad (6)$$

or

3) *there exist the differences f (x) Θ_H f (x + h), f (x) Θ_H f (x-h) and there also exists f '(x) ∈ I such that*

$$\lim_{h\to 0}\left(\frac{f(x)\theta_H f(x+h)}{-h}\right) = \lim_{h\to 0}\left(\frac{f(x)\theta_H f(x-h)}{h}\right) = f'(x) \quad (7)$$

or

4) *there exist the differences f (x + h) Θ_H f (x), f (x) Θ_H f (x-h) and there also exists f '(x) ∈ I such that*

$$\lim_{h\to 0}\left(\frac{f(x+h)\theta_H f(x)}{h}\right) = \lim_{h\to 0}\left(\frac{f(x)\theta_H f(x-h)}{-h}\right) = f'(x) \quad (8)$$

where the limits are taken in the metric space *(I, d_H)*.

As point out in [2], from an algebraic point of view, given two sets *A* and *B* in *I*, its difference can be expressed both in therms of addition or in therms of negative addition (that is, *C* is the difference between *A* and *B* if and only if *B = A + (-1)C*. This fact suggests a generalization of the H-difference, as introduced in [2], called the generalized Hukuhara difference of Aand *B* (gH-difference for short), denoted by $A\,\Theta_g\,B$ and defined as follows: for *A*, *B* ∈ *I*, the difference $A\,\Theta_g\,B$ is the element C ∈ I such that

$$A\,\theta_g B \iff \begin{Bmatrix} (a) & A = B + C, & or \\ (b) & B = A + (-1)C. \end{Bmatrix} \quad (9)$$

The first form (a) of (9) corresponds to the Hukuhara difference A Θ_H B. We denote to the second form (6) of (9) as $A \ominus_{g2}$ B

Proposition 1.3. *Let A, B ∈ I. Then*

i) $A \ominus_H B = -(B \ominus_{g2} A)$,

ii) $A \ominus_{g2} B = -(B \ominus_H A)$

Proposition 1.4.([2]). The g*H*-difference $C = A\,\Theta_g\,B$ of two intervals A = [a^L, a^U] and B = [b^L, b^U] always exists and

$$A\,\Theta_g\,B = \begin{cases} [a^L - b^L, a^U - b^U]\, if\ exist\ in\ the\ sense\ (a) of\ (9), or \\ [a^U - b^U, a^L - b^L]\, if\ exist\ in\ the\ sense\ (b) of\ (9). \end{cases}$$

Moreover, *if* $a^U - a^L = b^U - b^L$, then $A\,\Theta_g\,B$ = {c},where $c = a^U - b^U = a^L - b^L$

Remark 1.5. *Note that if* $a^U - a^L > b^U - b^L$, *then* $A\,\Theta_g\,B$ *exists in the sense (a) of (9); moreover if* $b^U - b^L > a^U - a^L$, *then* $A\,\Theta_g\,B$ *exists in the sense (b) of (9). In general a necessary condition to guarantee the existence of the gH-difference* $A\,\Theta_g\,B$, *for A, B ∈ I, is that* $A \supset B + \{c\}$ *or* $B \supset A + \{d\}$, *where A +* $\{d\}$ *and B + {c} , c, d ∈ ℝ, are the translations of A and B respectively.*

The generalized Hukuhara differences is used by Stefanini in [2] to introduce of concept generalized differentiability of a set-valued mapping.

Definition 1.6. ([2]). Let $f : T \to I$ *a set-valued mapping. Then f is said to be Hukuhara differentiable in a generalized sense (gH-differentiable for short) at a point* t_o *∈ T, if there exists differences* $f(t_o + h) \ominus_g f(t_o)$ *and there exists f '(*t_o *∈ I) such that*

$$f'(t_o) = \lim_{h\to 0}\frac{1}{h}\left[f(t_o + h) \ominus_g f(t_o)\right],$$

where the limit is taken in the space (*I*, d_H).

The notion of differentiability of Definition 1.6 is more much general that the notion of Definition 1.2.

Proposition 1.7. *If a set-valved function is generalized differentiability according to Definition 1.2, then it is gH-differentiable according to Definition 1.6.*

The following example shows that the implication in the proposition 1.7 is strict because the differences do not correspond with any differences to the forms 1, 2, 3, 4 of the Definition 1.2, hence the following set-valved mapping is gH-differentiable but it is not generalized differentiable.

Example 1.8. *Let* δ *a real function such that*

$$\delta(x) = \begin{cases} |x| & x \in \mathbb{Q}, \\ -|x| & x \notin \mathbb{Q}. \end{cases}$$

δ is not differentiability at zero, but if it is continuous at zero.

Let $f : \mathbb{R} \to I$ a set-valved function such that

$$f(x) = [-1 - \delta(t), 1 + \delta(t)]$$

Let $t_o = 0$. *We consider differential quotient*

$$\frac{f(t_0 + h) \ominus_g f(t_0)}{h} = \frac{f(h) \ominus_g f(0)}{h}$$

$$= \frac{[-1 - \delta(h), 1 + \delta(h)] \ominus_g [-1,1]}{h} =$$

$$= \begin{cases} \dfrac{[-1 - h, 1 + h] \ominus_{Hg} [-1,1]}{h}, & h \in \mathbb{Q}, \\ \dfrac{[-1 + h, 1 - h] \ominus_{g^2} [-1,1]}{h}, & h \notin \mathbb{Q} \end{cases}$$

$$= \begin{cases} \frac{[-1-h+1,1+h-1],}{h}, & h \in \mathbb{Q}, \\ \frac{[-1-h-1,-1+h+1]}{h}, & h \notin \mathbb{Q} \end{cases} =$$

$$= \frac{1}{h}[-h, h] = [-1,1] \rightarrow [-1,1], cuando\ h \rightarrow 0$$

Hence, the function f is gH-differentiable at t_0 *= 0.*

if we take two h_1, h_2 *positives and* $h_1 \in \mathbb{Q}, h_2 \notin \mathbb{Q}$, *then the differences f (t +* h_1*)* $\ominus_g$ *f (t) and f (t +* h_2*)* $\ominus_g$ *f (t) do no correspond with to differences of Definition 1.2.*

The following Proposition 1.9 is enounced in the paper [2] (Theorem 17) and it is used to obtain subsequent results. The previous Example 1.8 shows that it is false. Observe that *f* is gH-differentiable a$t_0 = 0$t, but real functions f^{L}(t) = -1- δ (t) and f^{U} (t) = 1 + δ (t) are not differentiates at $t_0 = 0$

Proposition 1.9. *f is gH-differentiable at x, then* f^L *and* f^U *are differentiable functions at x.*

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